

NONSINGULAR BLACK HOLES FROM GRAVITY–MATTER-BRANE LAGRANGIANS

EDUARDO GUENDELMAN* and ALEXANDER KAGANOVICH†

*Department of Physics, Ben-Gurion University of the Negev,
PO Box 653, IL-84105 Beer-Sheva, Israel*

**guendel@bgu.ac.il*

†*alexk@bgu.ac.il*

EMIL NISSIMOV‡ and SVETLANA PACHEVA§

*Institute for Nuclear Research and Nuclear Energy,
Bulgarian Academy of Sciences,
Boul. Tsarigradsko Chausee 72, BG-1784 Sofia, Bulgaria*

‡*nissimov@inrne.bas.bg*

§*svetlana@inrne.bas.bg*

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We consider self-consistent coupling of bulk Einstein–Maxwell–Kalb–Ramond system to codimension-one charged *lightlike* p -brane with dynamical (variable) tension (LL-brane). The latter is described by a manifestly reparametrization-invariant worldvolume action significantly different from the ordinary Nambu–Goto one. We show that the LL-brane is the appropriate gravitational and charge source in the Einstein–Maxwell–Kalb–Ramond equations of motion needed to generate a self-consistent solution describing *nonsingular* black hole. The latter consists of de Sitter interior region and exterior Reissner–Nordström region glued together along their common horizon (it is the inner horizon from the Reissner–Nordström side). The matching horizon is automatically occupied by the LL-brane as a result of its worldvolume Lagrangian dynamics, which dynamically generates the cosmological constant in the interior region and uniquely determines the mass and charge parameters of the exterior region. Using similar techniques we construct a self-consistent *wormhole* solution of Einstein–Maxwell system coupled to electrically neutral LL-brane, which describes two identical copies of a nonsingular black hole region being the exterior Reissner–Nordström region above the inner horizon, glued together along their common horizon (the inner Reissner–Nordström one) occupied by the LL-brane. The corresponding mass and charge parameters of the two black hole “universes” are explicitly determined by the dynamical LL-brane tension. This also provides an explicit example of Misner–Wheeler “charge without charge” phenomenon. Finally, this wormhole solution connecting two nonsingular black holes can be transformed into a special case of Kantowski–Sachs bouncing cosmology solution if instead of Reissner–Nordström we glue together two copies of the exterior Reissner–Nordström–de Sitter

region with big enough bare cosmological constant, such that the radial coordinate becomes a timelike variable everywhere in the two “universes,” except at the matching hypersurface occupied by the LL-brane.

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1. Introduction

The possibility of “*nonsingular* black holes” has been pointed out in the past few decades by numerous authors starting with the early works.^{1–3} The basic idea is to replace the standard black hole solutions in the region containing the singularity at the center of the geometry ($r = 0$) by regular solutions, which can be realized by introducing certain “vacuum-like matter” there.^{4–6} For a systematic recent review and an extensive list of references, see Ref. 7.

Several approaches to achieve the latter have been proposed in the literature which basically amount to:

- Regular solutions to Einstein equations smoothly interpolating between de Sitter-like behavior for small r and Schwarzschild-like, or Reissner–Nordström-like asymptotic behavior for large r . This can be achieved either by:
 - (a) choosing *a priori* an appropriate form for the matter energy–momentum tensor^{8–10} on the r.h.s. of Einstein equations without specifying a matter field Lagrangian from which it could be derived, or
 - (b) explicit construction of self-consistent solutions to Einstein equations, for gravity coupled to nonlinear electrodynamics,^{11–14} i.e. by deriving the pertinent energy–momentum tensor from a specific matter Lagrangian. As shown in Ref. 13, the latter includes the original Bardeen regular black hole solution.³

A classification scheme for the regular black hole solutions is provided in Ref. 15.

Recently a new approach for constructing regular black holes based on noncommutative geometry was advocated in Refs. 16–19. The singularity at $r = 0$ is removed and replaced by de Sitter core due to the short-distance fluctuations of the noncommutative coordinates.

Let us also note that similar regular black hole solutions with de Sitter behavior at small distances have been previously constructed in the context of integrable two-dimensional dilaton gravity.²⁰

- Matching of interior de Sitter with exterior Schwarzschild or Reissner–Nordström geometry across “thin shells,” where the shell dynamics is described in a non-Lagrangian way. The construction in the early works,^{6–23} where the matching occurs on the common horizon of the two geometries (i.e. matching along lightlike hypersurface), has been criticized in Refs. 24–26 (see also the no-go theorem for “false vacuum” black holes²⁷). More specifically Ref. 26 points out the

occurrence of discontinuity of the pressure in Refs. 21–23 across the horizon separating de Sitter and Schwarzschild regions. On the other hand, it has been shown (Ref. 28, p. 119) that interior de Sitter and exterior Reissner–Nordström regions can be consistently joined along common horizon without discontinuities in the metric and its first derivative (i.e. with no δ -function singularities in the Einstein equations), provided the common horizon is the inner horizon from the Reissner–Nordström side.

One should also mention the Frolov–Markov–Mukhanov model^{29,30} (see also Ref. 31) where de Sitter and Schwarzschild regions are matched along spacelike spherically symmetric surface layer.

In the present paper we will pursue a different approach aimed at deriving nonsingular black holes via matching of appropriate geometries along lightlike hypersurfaces in a self-consistent manner from first principles. Namely, we will explore the novel possibility of employing (charged) lightlike branes (LL-branes) (sometimes also called “null branes”) as natural self-consistent gravitational sources generating singularity-free black hole solutions from a well-defined *Lagrangian action principle* for bulk gravity–matter systems coupled to LL-branes. This is achieved:

- (a) through the appearance in the self-consistent bulk gravity–matter equations of motion of well-defined LL-brane stress–energy tensor derived from underlying reparametrization invariant worldvolume LL-brane action, as well as
- (b) through dynamical generation by the LL-brane of space–time varying cosmological constant.

The systematic Lagrangian action derivation of nonsingular black hole solutions through matching across LL-brane is the main feature which distinguishes our present approach from the previously proposed approaches using “thin shell” matching. In particular, as it will be shown below (Eqs. (62)) the problem with pressure discontinuity across the horizon pointed out in Ref. 26 does not appear in our approach.

LL-branes by themselves play an important role in general relativity as they enter the description of various physically important cosmological and astrophysical phenomena such as: (i) impulsive lightlike signals arising in cataclysmic astrophysical events;²⁸ (ii) the “membrane paradigm”³² of black hole physics; (iii) the thin-wall approach to domain walls coupled to gravity.^{33–35} More recently, LL-branes became significant also in the context of modern nonperturbative string theory, in particular, as the so-called *H*-branes describing quantum horizons (black hole and cosmological),³⁶ as Penrose limits of baryonic *D*-branes,³⁷ etc. (see also Refs. 38 and 39).

In the pioneering papers^{33–35} LL-branes in the context of gravity and cosmology have been extensively studied from a phenomenological point of view, i.e. by introducing them without specifying the Lagrangian dynamics from which they may

originate.^a On the other hand, we have proposed in a series of recent papers^{41–49} a new class of concise Lagrangian actions, providing a derivation from first principles of the LL-brane dynamics.

There are several characteristic features of LL-branes which drastically distinguish them from ordinary Nambu–Goto branes:

- (i) They describe intrinsically lightlike modes, whereas Nambu–Goto branes describe massive ones.
- (ii) The tension of the LL-brane arises as an *additional dynamical degree of freedom*, whereas Nambu–Goto brane tension is a given *ad hoc* constant. The latter characteristic feature significantly distinguishes our LL-brane models from the previously proposed *tensionless* p -branes (for a review, see Ref. 50) which rather resemble a p -dimensional continuous distribution of massless point-particles.
- (iii) Consistency of LL-brane dynamics in a spherically or axially symmetric gravitational background of codimension one requires the presence of an event horizon which is automatically occupied by the LL-brane (“horizon straddling” according to the terminology of Ref. 34).
- (iv) When the LL-brane moves as a *test* brane in spherically or axially symmetric gravitational backgrounds its dynamical tension exhibits exponential “inflation/deflation” time behavior^{45,46} — an effect similar to the “mass inflation” effect around black hole horizons.^{51,52}

A simple way to realize a nonsingular black hole is to consider an interior de Sitter space–time region matched to an exterior black hole region. In this context a particularly attractive mechanism for achieving such matching is provided by the LL-brane in view of the inherent “horizon straddling” property of its dynamics and the dynamical (variable) nature of its tension (properties (ii) and (iii) listed above).

The plan of the present paper is as follows. In Sec. 2 we consider self-consistent systems of bulk gravity and matter (Maxwell plus Kalb–Ramond gauge fields) interacting with (charged) LL-branes. On the way we briefly review our construction of reparametrization invariant LL-brane worldvolume actions for *arbitrary* worldvolume dimensions.

In Sec. 3 we discuss the properties of LL-brane equations of motion resulting from the LL-brane worldvolume action, including the couplings to the bulk electromagnetic and Kalb–Ramond gauge fields.

In Sec. 4 we consider the main features of LL-brane dynamics in spherically symmetric backgrounds, namely: (a) “horizon straddling,” (b) producing Coulomb field in the exterior region (above the horizon), and (c) dynamical generation of space–time varying bulk cosmological constant (acquiring different values below and above the horizon).

^aIn a more recent paper⁴⁰ brane actions in terms of their pertinent extrinsic geometry have been proposed which generically describe non-lightlike branes, whereas the lightlike branes are treated as a limiting case.

In Sec. 5 we present the explicit construction of a solution to the system of coupled Einstein–Maxwell–Kalb–Ramond–LL-brane equations with spherically symmetric geometry, which describes nonsingular black hole. The pertinent space–time manifold consists of two regions separated by a codimension-one lightlike hypersurface — the worldvolume of the LL-brane, which is simultaneously a common horizon of the interior region with de Sitter geometry and the exterior Reissner–Nordström (or Reissner–Nordström–de Sitter) region. Moreover, the LL-brane occupying the common horizon dynamically generates the cosmological constant in the interior region and uniquely determines the mass and charge parameters of the exterior region. Here a fundamental role is being played by the dynamical (variable) tension of the LL-brane, which in this particular solution turns out to vanish *on-shell*.

The resulting solution is indeed a nonsingular black hole although no violation of the weak energy condition occurs. This fact is in accordance with the observation in Refs. 53 and 54 that existence of nonsingular black holes is possible provided a topology change of the spacial sections takes place.

Section 6 is devoted to the construction of a self-consistent solution of Einstein–Maxwell system coupled to electrically neutral LL-brane, which describes *wormhole* connecting two nonsingular black hole regions (for an extensive general review of wormholes, see Ref. 55). The corresponding space–time manifold consists of two identical copies of the exterior Reissner–Nordström region above the *inner* Reissner–Nordström horizon, glued together by the LL-brane which automatically occupies their common horizon (the inner Reissner–Nordström one). The corresponding mass and charge parameters of the two Reissner–Nordström “universes” are explicitly determined by the dynamical LL-brane tension, which in this solution turns out to be strictly nonvanishing on-shell. The above wormhole solution also provides an explicit example of Misner–Wheeler “charge without charge” phenomenon.⁵⁶ Here, however, a violation of the null energy condition takes place (the LL-brane being an “exotic matter”) as predicted by general wormhole arguments (cf. Ref. 55).

In Sec. 7 the wormhole solution connecting two nonsingular black holes obtained in Sec. 6 is transformed into a cosmological bouncing solution — a special case of Kantowski–Sachs cosmology,⁵⁷ by considering Reissner–Nordström–de Sitter geometry with big enough bare cosmological constant, i.e. possessing one single horizon. Namely, the resulting space–time manifold consists of two identical copies of the exterior Reissner–Nordström–de Sitter space–time region (two identical “universes”) glued together along their common horizon occupied by the LL-brane, where the Reissner–Nordström–de Sitter radial coordinate becomes time-like coordinate in both “universes” except at the matching hypersurface (the LL-brane).

In the Appendix we describe in some detail the equivalence of LL-brane dynamics derived from the Nambu–Goto-type worldvolume action (Eq. (11) below) dual to the original Polyakov-type worldvolume action (Eq. (4) below).

2. Lagrangian Formulation of Einstein–Maxwell–Kalb–Ramond System Interacting with Lightlike Brane

Self-consistent bulk Einstein–Maxwell–Kalb–Ramond system coupled to a charged codimension-one *lightlike* p -brane (i.e. $D = (p + 1) + 1$) is described by the following action:

$$S = \int d^D x \sqrt{-G} \left[\frac{R(G)}{16\pi} - \frac{1}{4} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} - \frac{1}{D!2} \mathcal{F}_{\mu_1 \dots \mu_D} \mathcal{F}^{\mu_1 \dots \mu_D} \right] + \tilde{S}_{\text{LL}}. \quad (1)$$

Here $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$ and

$$\mathcal{F}_{\mu_1 \dots \mu_D} = D \partial_{[\mu_1} \mathcal{A}_{\mu_2 \dots \mu_D]} = \mathcal{F} \sqrt{-G} \varepsilon_{\mu_1 \dots \mu_D} \quad (2)$$

are the field-strengths of the electromagnetic \mathcal{A}_μ and Kalb–Ramond $\mathcal{A}_{\mu_1 \dots \mu_{D-1}}$ gauge potentials.^{58,59} The last term on the r.h.s. of (1) indicates the reparametrization invariant worldvolume action of the LL-brane coupled to the bulk gauge fields, proposed in our previous papers:^{41–49}

$$\begin{aligned} \tilde{S}_{\text{LL}} = S_{\text{LL}} - q \int d^{p+1} \sigma \varepsilon^{ab_1 \dots b_p} F_{b_1 \dots b_p} \partial_a X^\mu \mathcal{A}_\mu \\ - \frac{\beta}{(p+1)!} \int d^{p+1} \sigma \varepsilon^{a_1 \dots a_{p+1}} \partial_{a_1} X^{\mu_1} \dots \partial_{a_{p+1}} X^{\mu_{p+1}} \mathcal{A}_{\mu_1 \dots \mu_{p+1}}, \end{aligned} \quad (3)$$

where

$$S_{\text{LL}} = \int d^{p+1} \sigma \Phi \left[-\frac{1}{2} \gamma^{ab} g_{ab} + L(F^2) \right]. \quad (4)$$

In Eqs. (3), (4) the following notions and notations are used:

- Φ is alternative non-Riemannian integration measure density (volume form) on the p -brane worldvolume manifold:

$$\Phi \equiv \frac{1}{(p+1)!} \varepsilon^{a_1 \dots a_{p+1}} H_{a_1 \dots a_{p+1}}(B), \quad H_{a_1 \dots a_{p+1}}(B) = (p+1) \partial_{[a_1} B_{a_2 \dots a_{p+1}]} \quad (5)$$

instead of the usual $\sqrt{-\gamma}$. Here $\varepsilon^{a_1 \dots a_{p+1}}$ is the alternating symbol ($\varepsilon^{0^1 \dots p} = 1$), γ_{ab} ($a, b = 0, 1, \dots, p$) indicates the intrinsic Riemannian metric on the worldvolume, and $\gamma = \det \|\gamma_{ab}\|$. $H_{a_1 \dots a_{p+1}}(B)$ denotes the field-strength of an auxiliary worldvolume antisymmetric tensor gauge field $B_{a_1 \dots a_p}$ of rank p . As a special case one can build $H_{a_1 \dots a_{p+1}}$ in terms of $p + 1$ auxiliary worldvolume scalar fields $\{\varphi^I\}_{I=1}^{p+1}$:

$$H_{a_1 \dots a_{p+1}} = \varepsilon_{I_1 \dots I_{p+1}} \partial_{a_1} \varphi^{I_1} \dots \partial_{a_{p+1}} \varphi^{I_{p+1}}. \quad (6)$$

Note that γ_{ab} is *independent* of the auxiliary worldvolume fields $B_{a_1 \dots a_p}$ or φ^I . The alternative non-Riemannian volume form (5) has been first introduced in the context of modified standard (non-lightlike) string and p -brane models in Refs. 60 and 61.

- $X^\mu(\sigma)$ are the p -brane embedding coordinates in the bulk D -dimensional space–time with bulk Riemannian metric $G_{\mu\nu}(X)$ with $\mu, \nu = 0, 1, \dots, D - 1$; $(\sigma \equiv (\sigma^0 \equiv \tau, \sigma^i)$ with $i = 1, \dots, p$; $\partial_a \equiv \frac{\partial}{\partial \sigma^a}$.
- g_{ab} is the induced metric on worldvolume:

$$g_{ab} \equiv \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X), \tag{7}$$

which becomes *singular* on-shell (manifestation of the lightlike nature, cf. Eq. (20) below).

- $L(F^2)$ is the Lagrangian density of another auxiliary $(p - 1)$ -rank antisymmetric tensor gauge field $A_{a_1 \dots a_{p-1}}$ on the worldvolume with p -rank field-strength and its dual:

$$F_{a_1 \dots a_p} = p \partial_{[a_1} A_{a_2 \dots a_p]}, \quad F^{*a} = \frac{1}{p!} \frac{\varepsilon^{aa_1 \dots a_p}}{\sqrt{-\gamma}} F_{a_1 \dots a_p}. \tag{8}$$

$L(F^2)$ is *arbitrary* function of F^2 with the short-hand notation:

$$F^2 \equiv F_{a_1 \dots a_p} F_{b_1 \dots b_p} \gamma^{a_1 b_1} \dots \gamma^{a_p b_p}. \tag{9}$$

Rewriting the action (4) in the following equivalent form:

$$S = - \int d^{p+1} \sigma \chi \sqrt{-\gamma} \left[\frac{1}{2} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) - L(F^2) \right], \quad \chi \equiv \frac{\Phi}{\sqrt{-\gamma}} \tag{10}$$

with Φ the same as in (5), we find that the composite field χ plays the role of a *dynamical (variable) brane tension*. Let us note that the notion of dynamical brane tension has previously appeared in different contexts in Refs. 62–64.

Remark. It has been shown in Refs. 47–49 that the LL-brane equations of motion corresponding to the Polyakov-type action (4) (or (10)) can be equivalently obtained from the following *dual* Nambu–Goto-type action:

$$S_{\text{NG}} = - \int d^{p+1} \sigma T \sqrt{\left| \det \| g_{ab} - \epsilon \frac{1}{T^2} \partial_a u \partial_b u \| \right|}, \quad \epsilon = \pm 1. \tag{11}$$

Here T is *dynamical* tension simply proportional to the dynamical tension in the Polyakov-type formulation (4) ($T \sim \chi = \frac{\Phi}{\sqrt{-\gamma}}$), and u denotes the dual potential w.r.t. $A_{a_1 \dots a_{p-1}}$:

$$F_a^*(A) = \text{const} \frac{1}{\chi} \partial_a u. \tag{12}$$

Further details about the dynamical equivalence between the Polyakov-type and Nambu–Goto-type worldvolume Lagrangian formulation of LL-branes are contained in the Appendix. In the subsequent Secs. 3–5 we will consider the original Polyakov-type action (4).

The pertinent Einstein–Maxwell–Kalb–Ramond equations of motion derived from the action (1) read:

$$R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}R = 8\pi \left(T_{\mu\nu}^{(\text{EM})} + T_{\mu\nu}^{(\text{KR})} + T_{\mu\nu}^{(\text{brane})} \right), \tag{13}$$

$$\partial_\nu(\sqrt{-G}\mathcal{F}^{\mu\nu}) + q \int d^{p+1}\sigma \delta^{(D)}(x - X(\sigma)) \varepsilon^{ab_1 \dots b_p} F_{b_1 \dots b_p} \partial_a X^\mu = 0, \tag{14}$$

$$\varepsilon^{\nu\mu_1 \dots \mu_{p+1}} \partial_\nu \mathcal{F} - \beta \int d^{p+1}\sigma \delta^{(D)}(x - X(\sigma)) \varepsilon^{a_1 \dots a_{p+1}} \partial_{a_1} X^{\mu_1} \dots \partial_{a_{p+1}} X^{\mu_{p+1}} = 0, \tag{15}$$

where in the last equation we have used relation (2). The explicit form of the energy–momentum tensors read:

$$T_{\mu\nu}^{(\text{EM})} = \mathcal{F}_{\mu\kappa} \mathcal{F}_{\nu\lambda} G^{\kappa\lambda} - G_{\mu\nu} \frac{1}{4} \mathcal{F}_{\rho\kappa} \mathcal{F}_{\sigma\lambda} G^{\rho\sigma} G^{\kappa\lambda}, \tag{16}$$

$$\begin{aligned} T_{\mu\nu}^{(\text{KR})} &= \frac{1}{(D-1)!} \left[\mathcal{F}_{\mu\lambda_1 \dots \lambda_{D-1}} \mathcal{F}_{\nu}{}^{\lambda_1 \dots \lambda_{D-1}} - \frac{1}{2D} G_{\mu\nu} \mathcal{F}_{\lambda_1 \dots \lambda_D} \mathcal{F}^{\lambda_1 \dots \lambda_D} \right] \\ &= -\frac{1}{2} \mathcal{F}^2 G_{\mu\nu}, \end{aligned} \tag{17}$$

$$T_{\mu\nu}^{(\text{brane})} = -G_{\mu\kappa} G_{\nu\lambda} \int d^{p+1}\sigma \frac{\delta^{(D)}(x - X(\sigma))}{\sqrt{-G}} \chi \sqrt{-\gamma} \gamma^{ab} \partial_a X^\kappa \partial_b X^\lambda, \tag{18}$$

where the brane stress–energy tensor is straightforwardly derived from the world-volume action (4) (or, equivalently, (10); recall $\chi \equiv \frac{\Phi}{\sqrt{-\gamma}}$ is the variable brane tension).

Equations (14) and (15) show that:

- (i) the LL-brane is charged source for the bulk electromagnetism;
- (ii) the LL-brane uniquely determines the value of \mathcal{F}^2 in Eq. (17) through its coupling to the bulk Kalb–Ramond gauge field (Eq. (15)) which implies *dynamical generation* of bulk cosmological constant $\Lambda = 4\pi\mathcal{F}^2$.

The equations of motion of the LL-brane are discussed in some detail in the next section.

3. Lightlike Brane Dynamics

The equations of motion derived from the brane action (4) w.r.t. $B_{a_1 \dots a_p}$ are

$$\partial_a \left[\frac{1}{2} \gamma^{cd} g_{cd} - L(F^2) \right] = 0 \rightarrow \frac{1}{2} \gamma^{cd} g_{cd} - L(F^2) = M, \tag{19}$$

where M is an arbitrary integration constant. The equations of motion w.r.t. γ^{ab} read:

$$\frac{1}{2} g_{ab} - F^2 L'(F^2) \left[\gamma_{ab} - \frac{F_a^* F_b^*}{F^{*2}} \right] = 0, \tag{20}$$

where F^{*a} is the dual worldvolume field strength (8).

Remark 1. Before proceeding, let us mention that both the auxiliary worldvolume p -form gauge field $B_{a_1 \dots a_p}$ entering the non-Riemannian integration measure density (5), as well as the intrinsic worldvolume metric γ_{ab} are *nondynamical* degrees of freedom in the action (4), or equivalently, in (10). Indeed, there are no (time-) derivatives w.r.t. γ_{ab} , whereas the action (4) (or (10)) is *linear* w.r.t. the velocities $\partial_0 B_{a_1 \dots a_p}$. Thus, (4) is a constrained dynamical system, i.e. a system with gauge symmetries including the gauge symmetry under worldvolume reparametrizations, and both Eqs. (19), (20) are in fact *nondynamical constraint* equations (no second-order time derivatives present). Their meaning as constraint equations is best understood within the framework of the canonical Hamiltonian formalism for the action (4). Using the latter formalism one can also show that the auxiliary worldvolume $(p - 1)$ -form gauge field $A_{a_1 \dots a_{p-1}}$ (8) is nondynamical. The canonical Hamiltonian formalism for the action (4) can be developed in strict analogy with the Hamiltonian formalism for a simpler class of modified *non-lightlike* p -brane models based on the alternative non-Riemannian integration measure density (5), which was previously proposed in Ref. 65 (for details, we refer to Secs. 2 and 3 of Ref. 65). In particular, Eqs. (20) can be viewed as p -brane analogues of the string Virasoro constraints.

There are two important consequences of Eqs. (19), (20). Taking the trace in (20) and comparing with (19) implies the following crucial relation for the Lagrangian function $L(F^2)$:

$$L(F^2) - pF^2 L'(F^2) + M = 0, \tag{21}$$

which determines F^2 (9) on-shell as certain function of the integration constant M (19), i.e.

$$F^2 = F^2(M) = \text{const.} \tag{22}$$

The second and most profound consequence of Eqs. (20) is that the induced metric (7) on the worldvolume of the p -brane model (4) is *singular* on-shell (as opposed to the induced metric in the case of ordinary Nambu–Goto branes):

$$g_{ab} F^{*b} = 0, \tag{23}$$

i.e. the tangent vector to the worldvolume $F^{*a} \partial_a X^\mu$ is *lightlike* w.r.t. metric of the embedding space–time. Thus, we arrive at the following important conclusion: every point on the surface of the p -brane (4) moves with the speed of light in a time-evolution along the vector-field F^{*a} which justifies the name LL-brane (lightlike brane) model for (4).

Remark 2. Let us stress the importance of introducing the alternative non-Riemannian integration measure density in the form (5). If we would have started with worldvolume LL-brane action in the form (10) where the tension χ would be an *elementary* scalar field (instead of being a composite one — a ratio of two scalar densities as in the second relation in (10)), then variation w.r.t. χ would produce second Eq. (19) with M identically zero. This in turn by virtue of the constraint

(21) (with $M = 0$) would require the Lagrangian $L(F^2)$ to assume the special fractional power function form $L(F^2) = (F^2)^{1/p}$. In this special case the action (10) with elementary field χ becomes in addition manifestly invariant under *Weyl (conformal) symmetry*: $\gamma_{ab} \rightarrow \gamma'_{ab} = \rho \gamma_{ab}$, $\chi \rightarrow \chi' = \rho^{\frac{1-p}{2}} \chi$. This special case of Weyl-conformally invariant LL-branes has been discussed in our older papers (Refs. 41 and 42).

Let us point out that supplementing the LL-brane action (4) with natural couplings to bulk Maxwell and Kalb–Ramond gauge fields, as explicitly given by the worldvolume action in Eq. (3), does not affect Eqs. (19) and (20), so that the conclusions about on-shell constancy of F^2 (22) and the lightlike nature (23) of the p -branes under consideration remain unchanged. In what follows we will consider the extended LL-brane worldvolume action (3).

It remains to write down the equations of motion w.r.t. auxiliary worldvolume gauge field $A_{a_1 \dots a_{p-1}}$ and X^μ produced by the LL-brane action (3):

$$\partial_{[a} (F_{b]}^* \chi L'(F^2)) + \frac{q}{4} \partial_a X^\mu \partial_b X^\nu \mathcal{F}_{\mu\nu} = 0, \tag{24}$$

$$\begin{aligned} \partial_a (\chi \sqrt{-\gamma} \gamma^{ab} \partial_b X^\mu) + \chi \sqrt{-\gamma} \gamma^{ab} \partial_a X^\nu \partial_b X^\lambda \Gamma_{\nu\lambda}^\mu - q \varepsilon^{a b_1 \dots b_p} F_{b_1 \dots b_p} \partial_a X^\nu \mathcal{F}_{\lambda\nu} G^{\lambda\mu} \\ - \frac{\beta}{(p+1)!} \varepsilon^{a_1 \dots a_{p+1}} \partial_{a_1} X^{\mu_1} \dots \partial_{a_{p+1}} X^{\mu_{p+1}} \mathcal{F}_{\lambda\mu_1 \dots \mu_{p+1}} G^{\lambda\mu} = 0. \end{aligned} \tag{25}$$

Here χ is the dynamical brane tension as in (10), $\mathcal{F}_{\lambda\mu_1 \dots \mu_{p+1}}$ is the Kalb–Ramond field-strength (2),

$$\Gamma_{\nu\lambda}^\mu = \frac{1}{2} G^{\mu\kappa} (\partial_\nu G_{\kappa\lambda} + \partial_\lambda G_{\kappa\nu} - \partial_\kappa G_{\nu\lambda}) \tag{26}$$

is the Christoffel connection for the external metric, and $L'(F^2)$ denotes derivative of $L(F^2)$ w.r.t. the argument F^2 .

Worldvolume reparametrization invariance allows us to introduce the standard synchronous gauge-fixing conditions:

$$\gamma^{0i} = 0 \quad (i = 1, \dots, p), \quad \gamma^{00} = -1. \tag{27}$$

Also, we will use a natural ansatz for the “electric” part of the auxiliary worldvolume gauge field-strength (8):

$$F^{*i} = 0 \quad (i = 1, \dots, p) \quad \text{i.e.} \quad F_{0i_1 \dots i_{p-1}} = 0, \tag{28}$$

meaning that we choose the lightlike direction in Eq. (23) to coincide with the brane proper-time direction on the worldvolume ($F^{*a} \partial_a \sim \partial_\tau$). The Bianchi identity ($\nabla_a F^{*a} = 0$) together with (27), (28) and the definition for the dual field-strength in (8) imply:

$$\partial_0 \gamma^{(p)} = 0, \quad \text{where} \quad \gamma^{(p)} \equiv \det \|\gamma_{ij}\|. \tag{29}$$

Then the LL-brane equations of motion acquire the form (recall definition of the induced metric g_{ab} (7)):

$$g_{00} \equiv \dot{X}^\mu G_{\mu\nu} \dot{X}^\nu = 0, \quad g_{0i} = 0, \quad g_{ij} - 2a_0 \gamma_{ij} = 0 \tag{30}$$

(the latter are analogs of Virasoro constraints in standard string theory), where a_0 is a M -dependent constant:

$$a_0 \equiv F^2 L'(F^2)|_{F^2=F^2(M)}; \tag{31}$$

$$\partial_i \chi + \frac{q}{a_1} \partial_0 X^\mu \partial_i X^\nu \mathcal{F}_{\mu\nu} = 0, \quad \partial_i X^\mu \partial_j X^\nu \mathcal{F}_{\mu\nu} = 0, \tag{32}$$

with

$$a_1 \equiv 2 \sqrt{\frac{F^2}{p!}} L'(F^2) \Big|_{F^2=F^2(M)} = \text{const}, \tag{33}$$

$$\begin{aligned} & -\sqrt{\gamma^{(p)}} \partial_0 (\chi \partial_0 X^\mu) + \partial_i (\chi \sqrt{\gamma^{(p)}} \gamma^{ij} \partial_j X^\mu) \\ & + \chi \sqrt{\gamma^{(p)}} (-\partial_0 X^\nu \partial_0 X^\lambda + \gamma^{kl} \partial_k X^\nu \partial_l X^\lambda) \Gamma_{\nu\lambda}^\mu - q \sqrt{p! F^2} \sqrt{\gamma^{(p)}} \partial_0 X^\nu \mathcal{F}_{\lambda\nu} G^{\lambda\mu} \\ & - \frac{\beta}{(p+1)!} \mathcal{F} \varepsilon^{a_1 \dots a_{p+1}} \partial_{a_1} X^{\mu_1} \dots \partial_{a_{p+1}} X^{\mu_{p+1}} \varepsilon_{\lambda\mu_1 \dots \mu_{p+1}} \sqrt{-G} G^{\lambda\mu} = 0. \end{aligned} \tag{34}$$

Let us recall that $F^2 = \text{const}$ (Eq. (22)).

4. Lightlike Brane in Spherically Symmetric Backgrounds

In what follows we will be interested in static spherically symmetric solutions to Einstein–Maxwell–Kalb–Ramond equations (13)–(15). The generic form of spherically symmetric metric in Eddington–Finkelstein coordinates^{66,67} reads

$$ds^2 = -A(r)dv^2 + 2dv dr + C(r)h_{ij}(\theta)d\theta^i d\theta^j, \tag{35}$$

where h_{ij} indicates the standard metric on S^p . We will consider the simplest ansatz for the LL-brane embedding coordinates:

$$X^0 \equiv v = \tau, \quad X^1 \equiv r = r(\tau), \quad X^i \equiv \theta^i = \sigma^i \quad (i = 1, \dots, p). \tag{36}$$

Now, the LL-brane equations (30) together with (29) yield

$$-A(r) + 2\dot{r} = 0, \quad \partial_\tau C = \dot{r} \partial_r C|_{r=r(\tau)} = 0, \tag{37}$$

implying

$$\dot{r} = 0 \rightarrow r = r_0 = \text{const}, \quad A(r_0) = 0. \tag{38}$$

Equation (38) tells us that consistency of LL-brane dynamics in a spherically symmetric gravitational background of codimension one requires the latter to possess a horizon (at some $r = r_0$), which is automatically occupied by the LL-brane (“horizon straddling” according to the terminology of Ref. 34). Similar property —

“horizon straddling,” has also been found for LL-branes moving in rotating axially symmetric (Kerr or Kerr–Newman) and rotating cylindrically symmetric black hole backgrounds (Refs. 47 and 49).

Next, the Maxwell coupling of the LL-brane produces via Eq. (14) static Coulomb field in the outer region beyond the horizon. Namely, inserting in Eq. (14) the embedding ansatz (36) together with (38) and accounting for (27)–(30) we obtain

$$\partial_r(C^{p/2}(r)\mathcal{F}_{vr}(r)) - q \frac{\sqrt{p!F^2}}{(2a_0)^{p/2}}C^{p/2}(r_0)\delta(r - r_0) = 0, \tag{39}$$

which yields for the Maxwell field-strength:

$$\mathcal{F}_{vr}(r) = \left(\frac{C(r_0)}{C(r)}\right)^{p/2} \frac{q\sqrt{p!F^2}}{(2a_0)^{p/2}}\theta(r - r_0). \tag{40}$$

Taking into account the Coulomb form of the bulk Maxwell field-strength (40) together with (36) and (38), we obtain that LL-brane equations (32) reduce to simply $\partial_i\chi = 0$.

Using again the embedding ansatz (36) together with (38) as well as (27)–(30), the Kalb–Ramond equations of motion (15) reduce to

$$\partial_r\mathcal{F} + \beta\delta(r - r_0) = 0 \rightarrow \mathcal{F} = \mathcal{F}_{(+)}\theta(r - r_0) + \mathcal{F}_{(-)}\theta(r_0 - r), \tag{41}$$

$$\mathcal{F}_{(\pm)} = \text{const}, \quad \mathcal{F}_{(-)} - \mathcal{F}_{(+)} = \beta. \tag{42}$$

Therefore, a space–time varying nonnegative cosmological constant is dynamically generated in both exterior and interior regions w.r.t. the horizon at $r = r_0$ (cf. Eq. (17)):

$$\Lambda_{(\pm)} = 4\pi\mathcal{F}_{(\pm)}^2. \tag{43}$$

Finally, it remains to consider the second order (w.r.t. proper time derivative) X^μ equations of motion (34). Upon inserting the embedding ansatz (36) together with (38) and taking into account (30), (40) and (42), we find that the only non-trivial equations is for $\mu = v$. Before proceeding let us note that the “force” terms in the X^μ equations of motion (34) (the geodesic ones containing the Christoffel connection coefficients as well as those coming from the LL-brane coupling to the bulk Maxwell and Kalb–Ramond gauge fields) contain discontinuities across the horizon occupied by the LL-brane. The discontinuity problem is resolved following the approach in Ref. 33 (see also the regularization approach in Ref. 68, App. A) by taking mean values of the “force” terms across the discontinuity at $r = r_0$. Thus, we obtain from Eq. (34) with $\mu = v$:

$$\begin{aligned} \partial_r\chi + \chi \left[\frac{1}{4}(\partial_r A_{(+)} + \partial_r A_{(-)}) + pa_0\partial_r \ln C \right]_{r=r_0} \\ + \frac{1}{2} \left[-q^2 \frac{p!F^2}{(2a_0)^{p/2}} + \beta(2a_0)^{p/2}(\mathcal{F}_{(-)} + \mathcal{F}_{(+)}) \right] = 0. \end{aligned} \tag{44}$$

5. Nonsingular Black Hole Solution

Let us go back to the Einstein equations of motion (13), where the LL-brane energy–momentum tensor (18), upon inserting the expressions for $X^\mu(\sigma)$ from (36) and (38), and taking into account (27), (29) and (30), acquires the form:

$$T_{(\text{brane})}^{\mu\nu} = S^{\mu\nu} \delta(r - r_0) \tag{45}$$

with surface energy–momentum tensor:

$$S^{\mu\nu} \equiv \frac{\chi}{(2a_0)^{p/2}} [\partial_\tau X^\mu \partial_\tau X^\nu - 2a_0 G^{ij} \partial_i X^\mu \partial_j X^\nu]_{v=\tau, r=r_0, \theta^i=\sigma^i} . \tag{46}$$

Here again a_0 is the integration constant parameter appearing in the LL-brane dynamics (31) and $G_{ij} = C(r)h_{ij}(\theta)$. For the nonzero components of $S_{\mu\nu}$ (with lower indices) and its trace we find:

$$S_{rr} = \frac{\chi}{(2a_0)^{p/2}} , \quad S_{ij} = -\frac{\chi}{(2a_0)^{p/2-1}} G_{ij} , \quad S_\lambda^\lambda = -\frac{p\chi}{(2a_0)^{p/2-1}} . \tag{47}$$

The solutions to the other bulk space–time equations of motion (the Maxwell (14) and Kalb–Ramond (15)) with spherically symmetric geometry have already been given in the previous section, see Eqs. (39)–(42).

For the sake of simplicity we will consider in what follows the case of ($D = 4$)-dimensional bulk space–time and, correspondingly, $p = 2$ for the LL-brane. The generalization to arbitrary D is straightforward. For further simplification of the numerical constant factors we will choose the following specific form for the Lagrangian of the auxiliary nondynamical worldvolume gauge field (cf. Eqs. (8), (9)):

$$L(F^2) = \frac{1}{4} F^2 \rightarrow a_0 = M , \tag{48}$$

where a_0 is the constant defined in (31) and M denotes the original integration constant in Eqs. (19) and (21).

We will show that there exists a spherically symmetric solution of the Einstein equations of motion (13) with LL-brane energy–momentum tensor on the r.h.s. given by (46), (47) — systematically derived from the reparametrization invariant LL-brane worldvolume action (4), which describes a *nonsingular* black hole. Namely, this solutions describes a space–time consisting of two spherically symmetric regions — an interior de Sitter region (for $r < r_0$) and an exterior Reissner–Nordström (or Reissner–Nordström–de Sitter) region (for $r > r_0$) matched along common horizon at $r = r_0$, where r_0 is the *inner* horizon from the Reissner–Nordström side and which is automatically occupied by the LL-brane (“horizon straddling”) as shown in Eqs. (37), (38) above. Moreover:

- (a) The surface charge density q of the LL-brane (cf. (14) and (39)) explicitly determines the nonzero Coulomb field-strength in the exterior region $r > r_0$ (Eq. (40)) so that the Reissner–Nordström charge parameter is given explicitly by (for $D = 4$ space–time dimensions):

$$Q^2 = \frac{8\pi}{a_0} q^2 r_0^4 . \tag{49}$$

(b) As shown in Eqs. (41)–(43), the LL-brane through its coupling to the bulk Kalb–Ramond field (cf. (3) and (15)) dynamically generates space–time varying nonnegative cosmological constant with a jump across the horizon ($r = r_0$). In particular, we will consider the case of vanishing cosmological constant in the exterior region ($r > r_0$) (pure Reissner–Nordström geometry) which leaves a dynamically generated de Sitter geometry below the horizon ($r < r_0$), i.e. $\Lambda_{(-)} = 4\pi\beta^2$.

In Eddington–Finkelstein coordinates the nonsingular black hole is given by

$$ds^2 = -A(r)dv^2 + 2dv dr + r^2 h_{ij}(\theta)d\theta^i d\theta^j, \tag{50}$$

$$A(r) \equiv A_{(-)}(r) = 1 - \frac{4\pi}{3} \mathcal{F}_{(-)}^2 r^2, \quad \text{for } r < r_0, \tag{51}$$

$$A(r) \equiv A_{(+)}(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{4\pi}{3} \mathcal{F}_{(+)}^2 r^2, \quad \text{for } r > r_0, \tag{52}$$

where $\mathcal{F}_{(\pm)}$ and Q^2 is given by (42) and (49), respectively, and where r_0 is the common horizon:

$$A_{(-)}(r_0) = 0, \quad A_{(+)}(r_0) = 0, \quad r_0 = m - \sqrt{m^2 - Q^2} \quad (\text{for } \mathcal{F}_{(+)} = 0). \tag{53}$$

Since the above metrics obviously solve Einstein equations (13) in their respective interior ($r < r_0$) and exterior ($r > r_0$) regions, it only remains to study their matching along the common horizon ($r = r_0$).

The systematic formalism for matching different bulk space–time geometries on codimension-one timelike hypersurfaces (“thin shells”) was developed originally in Ref. 33 and later generalized in Ref. 34 to the case of lightlike hypersurfaces (“null thin shells”) (for a systematic introduction, see the textbook Ref. 69). In the present case, due to the simple geometry (spherical symmetry and matching on common horizon) one can straightforwardly isolate the terms from the Ricci tensor on the l.h.s. of Einstein equations (13) which may yield delta-function contributions ($\sim \delta(r - r_0)$) to be matched with the components of the LL-brane surface stress–energy tensor (45), (46).

Since the metric (50)–(52) is continuous at $r = r_0$ (due to Eq. (53)), but its first derivative w.r.t. r (the normal coordinate w.r.t. horizon) might exhibit discontinuity across $r = r_0$, the terms contributing to δ -function singularities in $R_{\mu\nu}$ are those containing second derivatives w.r.t. r . Separating explicitly the latter we can rewrite Eqs. (13) in the following form (here we take $D = 4$ and $p = 2$):

$$\begin{aligned} R_{\mu\nu} &\equiv \partial_r \Gamma_{\mu\nu}^r - \partial_\mu \partial_\nu \ln \sqrt{-G} + \text{nonsingular terms} \\ &= 8\pi \left(S_{\mu\nu} - \frac{1}{2} G_{\mu\nu} S_\lambda^\lambda \right) \delta(r - r_0) + \text{nonsingular terms}. \end{aligned} \tag{54}$$

For $(\mu, \nu) = (r, r)$ the r.h.s. of Eq. (54) has nonzero δ -function contribution due to the nonzero component $S_{rr} = \frac{\chi}{2a_0}$ (cf. Eqs. (47)) while on the l.h.s. $\Gamma_{rr}^r = 0$ for the metric (50) and $\sqrt{-G} = r^2$ is a smooth function across $r = r_0$, i.e.

there is no $\delta(r - r_0)$ contribution on the l.h.s. Therefore, the matching dictates that the dynamical LL-brane tension χ must vanish on-shell in the solution under consideration.

Then, Eq. (54) for $(\mu, \nu) = (v, r)$ yields

$$\partial_r A_{(+)}|_{r=r_0} - \partial_r A_{(-)}|_{r=r_0} = 0. \tag{55}$$

Let us note that in spite of the continuity of the metric (50) and its first derivative w.r.t. r across the horizon ($r = r_0$) (Eqs. (53) and (55)), the matching is *not* completely smooth since the higher derivatives of (50) are discontinuous there.

Finally, taking into account that $\chi = 0$ on-shell, Eq. (44) yields a second relation between the dynamically generated cosmological constants below and above the horizon (here $p = 2$):

$$\mathcal{F}_{(-)} + \mathcal{F}_{(+)} = \frac{2q^2}{a_0\beta}, \tag{56}$$

with together with (42) completely determines $\mathcal{F}_{(\pm)}$ as functions of the LL-brane charge and Kalb–Ramond coupling parameters:

$$\mathcal{F}_{(\pm)} = \frac{q^2}{a_0\beta} \mp \frac{\beta}{2}. \tag{57}$$

Solving the matching equations (53), (55) we get

$$r_0 = \frac{1}{\sqrt{K_{(-)}}}, \quad m = \frac{2}{\sqrt{K_{(-)}}} \left(1 - \frac{K_{(+)}}{K_{(-)}} \right), \quad Q^2 = \frac{3}{K_{(-)}} \left(1 - \frac{K_{(+)}}{K_{(-)}} \right), \tag{58}$$

where (cf. Eqs. (57))

$$K_{(\pm)} \equiv \frac{4\pi}{3} \mathcal{F}_{(\pm)}^2 = \frac{4\pi}{3} \left(\frac{q^2}{a_0\beta} \mp \frac{\beta}{2} \right)^2, \tag{59}$$

and the expression for Q^2 in (58) is identical to that in Eq. (49).

In particular, choosing the LL-brane integration constant $a_0 = 2\frac{q^2}{\beta^2}$ we have from (57):

$$\mathcal{F}_- = \beta, \quad \mathcal{F}_+ = 0, \tag{60}$$

i.e. vanishing dynamical cosmological constant above horizon. In this special case of interior de Sitter region matched to pure Reissner–Nordström exterior region Eqs. (58) simplify to the followings:

$$r_0 = \frac{1}{\sqrt{K}}, \quad m = \frac{2}{\sqrt{K}}, \quad Q^2 = \frac{3}{K}, \quad \text{with } K = \frac{4\pi}{3}\beta^2. \tag{61}$$

It is now straightforward to check the *absence* of pressure discontinuity across the matching horizon. Indeed, using the relations (40) and (43), (57) for the LL-brane-generated exterior Coulomb field-strength and space–time-varying cosmological constant and inserting them into the expressions for the corresponding

energy–momentum tensors (16), (17) we obtain for the mixed diagonal $\binom{r}{r}$ components of the latter at the common horizon:

$$(T_{\text{dS}}^{\text{interior}})^r_r = (T_{\text{RN+dS}}^{\text{exterior}})^r_r = -\frac{1}{2} \left(\frac{q^2}{a_0\beta} + \frac{\beta}{2} \right)^2. \quad (62)$$

Similar conclusion about absence of discontinuity in the energy density across the matching horizon is true as well, since in the present case $T_0^0 = T_r^r$ in both regions.

6. Wormhole Connecting Two Nonsingular Black Holes

Let us now consider a self-consistent bulk Einstein–Maxwell system (*without* cosmological constant) free of electrically charged matter, coupled to a electrically neutral LL-brane:

$$S = \int d^D x \sqrt{-G} \left[\frac{R(G)}{16\pi} - \frac{1}{4} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} \right] + S_{\text{LL}}, \quad (63)$$

where S_{LL} is the same as in (4) (Polyakov-type LL-brane action) or in (A.1) (Nambu–Goto-type LL-brane action). The pertinent Einstein–Maxwell equations of motion read:

$$R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R = 8\pi \left(T_{\mu\nu}^{(\text{EM})} + T_{\mu\nu}^{(\text{brane})} \right), \quad \partial_\nu (\sqrt{-G} G^{\mu\kappa} G^{\nu\lambda} \mathcal{F}_{\kappa\lambda}) = 0, \quad (64)$$

where $T_{\mu\nu}^{(\text{EM})}$ is given by (16) and $T_{\mu\nu}^{(\text{brane})}$ is the same as in (18) or, equivalently, (A.12).

Using the general formalism for constructing self-consistent spherically symmetric or rotating cylindrical *wormhole* solutions via LL-branes presented in Ref. 49, we will now construct traversable (w.r.t. the proper time of traveling observer, see below) wormhole solution to the Einstein equations (64) which:

- (a) will connect two identical nonsingular black hole space–time regions — two copies of the exterior Reissner–Nordström region above the *inner* Reissner–Nordström horizon;
- (b) will combine the features of the Einstein–Rosen “bridge” in its original formulation⁷⁰ (wormhole throat at horizon)^b and the feature “charge without charge” of Misner–Wheeler wormholes.⁵⁶

^bLet us particularly emphasize that the Einstein–Rosen “bridge” in its original formulation⁷⁰ is a four-dimensional space–time manifold consisting of two copies of the exterior Schwarzschild space–time region matched along the horizon. On the other hand, the nomenclature of “Einstein–Rosen bridge” in several standard textbooks (e.g. Ref. 71) uses the Kruskal–Szekeres manifold and it is *not equivalent* to the original construction in Ref. 70. Namely, the two regions in Kruskal–Szekeres space–time corresponding to the outer Schwarzschild space–time region ($r > 2m$) and labeled (I) and (III) in Ref. 71 are generally *disconnected* and share only a two-sphere (the angular part) as a common border ($U = 0, V = 0$ in Kruskal–Szekeres coordinates), whereas in the original Einstein–Rosen “bridge” construction the boundary between the two identical copies of the outer Schwarzschild space–time region ($r > 2m$) is a three-dimensional hypersurface ($r = 2m$). For discussion on the role of LL-brane as matter source of the original Einstein–Rosen “bridge” wormhole, see Ref. 78.

In doing this we will follow the standard procedure described in Ref. 55, but with the significant difference that in our case we will solve Einstein equations following from a well-defined Lagrangian action ((63) with (4) or (A.1)) describing self-consistent bulk gravity–matter system coupled to a LL-brane. In other words, the LL-brane will serve as a gravitational source of the wormhole by locating itself on its throat as a result of its consistent worldvolume dynamics (Eq. (38) above).

Let us introduce the following modification of the standard Reissner–Nordström metric in Eddington–Finkelstein coordinates:

$$ds^2 = -\tilde{A}(\eta)dv^2 + 2dv d\eta + \tilde{r}^2(\eta)h_{ij}(\vec{\theta})d\theta^i d\theta^j, \tag{65}$$

$$\tilde{A}(\eta) \equiv A(\tilde{r}(\eta)) = 1 - \frac{2m}{\tilde{r}(\eta)} + \frac{Q^2}{\tilde{r}^2(\eta)}, \quad \tilde{r}(\eta) = r_{(-)} + |\eta|, \tag{66}$$

where

$$r_{(-)} \equiv m - \sqrt{m^2 - Q^2}, \quad -\infty < \eta < \infty. \tag{67}$$

From now on the bulk space–time indices μ, ν will refer to (v, η, θ^i) instead of (v, r, θ^i) . The new metric (65), (66) represents two identical copies of the exterior Reissner–Nordström space–time region ($r > r_{(-)}$), which are sewed together along the internal Reissner–Nordström horizon ($r = r_{(-)}$) (67). We will show that the new metric (65), (66) is a solution of the Einstein equations (64) thanks to the presence of $T_{\mu\nu}^{(\text{brane})}$ on the r.h.s. Here the newly introduced coordinate η will play the role of a radial-like coordinate normal w.r.t. the LL-brane located on the throat, which interpolates between the two Reissner–Nordström “universes” for $\eta > 0$ and $\eta < 0$ (the two copies transform into each other under the “parity” transformation $\eta \rightarrow -\eta$). Each of these Reissner–Nordström “universes” represents a nonsingular black hole space–time region since they still contain horizons at $\eta = 2\sqrt{m^2 - Q^2}$ and $\eta = -2\sqrt{m^2 - Q^2}$, respectively, which correspond to the outer horizon $r_{(+)} \equiv m + \sqrt{m^2 - Q^2}$ of the standard Reissner–Nordström metric.

As in (36) we will use the simplest embedding for the LL-brane coordinates $(X^\mu) \equiv (v, \eta, \theta^i) = (\tau, \eta(\tau), \sigma^i)$. In complete analogy with (37), (38) we find that the LL-brane equations of motion following from the underlying worldvolume action (4) or, equivalently, (11) yield $\eta(\tau) = 0$, i.e. the LL-brane automatically locates itself on the junction ($\eta = 0$) between the two Reissner–Nordström “universes.” The pertinent LL-brane stress–energy tensor (46) or, equivalently, (A.13):

$$T_{(\text{brane})}^{\mu\nu} = S^{\mu\nu} \delta(\eta), \tag{68}$$

$$S^{\mu\nu} \equiv \frac{T}{\epsilon\sqrt{b_0}} [\partial_\tau X^\mu \partial_\tau X^\nu - \epsilon b_0 G^{ij} \partial_i X^\mu \partial_j X^\nu]_{v=\tau, \eta=0, \theta^i=\sigma^i}$$

has the following nonzero components with lower indices and trace (hereafter we will consider the special case $p = 2$ for simplicity):

$$S_{\eta\eta} = \epsilon \frac{T}{\sqrt{b_0}}, \quad S_{ij} = -T\sqrt{b_0}G_{ij}, \quad S_\lambda^\lambda = -2T\sqrt{b_0}. \tag{69}$$

Let us now turn to the Einstein equations (64) where we explicitly separate the terms contributing to δ -function singularities $\sim \delta(\eta)$ on the l.h.s. These are the terms containing second-order derivatives w.r.t. η , since the metric coefficients in (65), (66) are functions of $|\eta|$ and $\partial_\eta^2|\eta| = 2\delta(\eta)$. Thus, we have

$$\begin{aligned}
 R_{\mu\nu} &\equiv \partial_\eta \Gamma_{\mu\nu}^\eta - \partial_\mu \partial_\nu \ln \sqrt{-G} + \text{nonsingular terms} \\
 &= 8\pi \left(S_{\mu\nu} - \frac{1}{2} G_{\mu\nu} S_\lambda^\lambda \right) \delta(\eta) + \text{nonsingular terms}, \tag{70}
 \end{aligned}$$

with the LL-brane stress–energy given by (69). Using the explicit expressions for the Christoffel coefficients:

$$\Gamma_{\nu\nu}^\eta = \frac{1}{2} \tilde{A} \partial_\eta \tilde{A}, \quad \Gamma_{v\eta}^\eta = -\frac{1}{2} \partial_\eta \tilde{A}, \quad \Gamma_{ij}^\eta = -\frac{1}{2} \tilde{A} G_{ij} \partial_\eta \ln \tilde{r}^2, \quad \sqrt{-G} = \tilde{r}^2 \tag{71}$$

with $\tilde{A}(\eta)$ and $\tilde{r}(\eta)$ as in (66) and taking into account $\tilde{A}(0) = 0$, it is straightforward to check that nonzero δ -function contributions in $R_{\mu\nu}$ appear for $(\mu\nu) = (v\eta)$ and $(\mu\nu) = (\eta\eta)$ only. Matching the coefficients in front of the $\delta(\eta)$ on both sides of (70) yields accordingly (here for consistency we have to choose $\epsilon = -1$ in (69)):

$$4\pi T \sqrt{b_0} r_{(-)}^2 + r_{(-)} - m = 0, \quad r_{(-)} = \frac{\sqrt{b_0}}{2\pi T}, \tag{72}$$

with $r_{(-)}$ as in (67) (radius of the inner Reissner–Nordström horizon). From (72) we obtain the following expressions for the mass and charge parameters of the Reissner–Nordström “universes” as functions of T (the LL-brane dynamical tension) and the free parameter b_0 appearing in the LL-brane dynamics (cf. Appendix):

$$m = \frac{\sqrt{b_0}}{2\pi T} (1 + 2b_0), \quad Q^2 = \frac{b_0}{(2\pi T)^2} (1 + 4b_0). \tag{73}$$

The wormhole connecting two nonsingular Reissner–Nordström black hole “universes” constructed above is *traversable* w.r.t. the *proper-time* of in-falling particles (“traveling observers”). This can be directly inferred from the equation for point-particle motion (cf. Ref. 72, Chap. 8, Sec. 4) along η — “radial geodesics”:

$$\eta'^2 + \mathcal{V}_{\text{eff}}(\eta) = \frac{\mathcal{E}^2}{m_0^2}, \quad \mathcal{V}_{\text{eff}}(\eta) \equiv \tilde{A}(\eta) \left(1 + \frac{\mathcal{J}^2}{m_0^2 \tilde{r}^2(\eta)} \right), \tag{74}$$

where the prime indicates proper-time derivative, $\tilde{A}(\eta)$ and $\tilde{r}(\eta)$ are the same as in (66), and m_0 , \mathcal{E} and \mathcal{J} denote the particle mass, its conserved energy and conserved angular momentum. The form of the “effective potential” $\mathcal{V}_{\text{eff}}(\eta)$ for $\mathcal{J} = 0$ is depicted in Fig. 1.

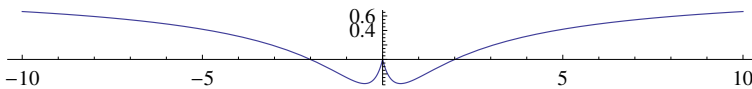


Fig. 1. Shape of $\mathcal{V}_{\text{eff}}(\eta)$ as a function of the dimensionless ratio $x \equiv \eta/r_{(-)}$.

However, in accordance with the general arguments⁵⁵ the LL-brane at the wormhole throat represents an “exotic matter” since its stress–energy tensor (68), (69) violates the null-energy condition.

7. Bouncing Cosmology via Lightlike Brane

Finally, let us consider an extension of the model (63) describing Einstein–Maxwell system interacting with a LL-brane by adding a *bare* cosmological constant term:

$$S = \int d^D x \sqrt{-G} \left[\frac{R(G)}{16\pi} - \frac{\Lambda}{8\pi} - \frac{1}{4} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} \right] + S_{\text{LL}}. \tag{75}$$

Accordingly, the Einstein equations of motion read

$$R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R + \Lambda G_{\mu\nu} = 8\pi \left(T_{\mu\nu}^{(\text{EM})} + T_{\mu\nu}^{(\text{brane})} \right), \tag{76}$$

using the same notations as in (63), (64).

Equations (76) in the absence of the LL-brane stress–energy tensor possess spherically symmetric Reissner–Nordström–de Sitter solution (here again we consider $D = 4$):

$$ds^2 = -A_\Lambda(r) dt^2 + \frac{dr^2}{A_\Lambda(r)} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \tag{77}$$

or, written in Eddington–Finkelstein coordinates ($dt = dv - \frac{dr}{A_\Lambda(r)}$):

$$ds^2 = -A_\Lambda(r) dv^2 + 2dv dr + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \tag{78}$$

$$A_\Lambda(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3} r^2.$$

For sufficiently large Λ there is only one *single* horizon at $r = r_{(0)} \equiv r_{(0)}(m, Q^2, \Lambda)$, where

$$A_\Lambda(r_{(0)}) = 0, \quad \partial_r A_\Lambda|_{r=r_{(0)}} < 0, \quad \text{i.e.} \quad A_\Lambda(r) < 0 \quad \text{for} \quad r > r_{(0)}. \tag{79}$$

Let us now consider the following modification of the Reissner–Nordström–de Sitter metric (78):

$$ds^2 = \tilde{A}_\Lambda(\eta) dv^2 + 2dv d\eta + \tilde{r}^2(\eta) (d\theta^2 + \sin^2 \theta d\varphi^2), \tag{80}$$

$$\tilde{A}_\Lambda(\eta) \equiv -A_\Lambda(r_{(0)} + |\eta|) = \frac{\Lambda}{3} \tilde{r}^2(\eta) - \frac{Q^2}{\tilde{r}^2(\eta)} + \frac{2m}{\tilde{r}(\eta)} - 1, \tag{81}$$

where $\tilde{r}(\eta) = r_{(0)} + |\eta|$ and as above η varies from $-\infty$ to $+\infty$. Because of (79) the metric component \tilde{A}_Λ (81) in (80) is always nonnegative:

$$\tilde{A}_\Lambda(\eta) > 0 \quad \text{for} \quad \eta > 0 \quad \text{or} \quad \eta < 0; \quad \tilde{A}_\Lambda(0) = 0. \tag{82}$$

The metric (80), (81) corresponds to a space–time manifold consisting of two identical copies of the *exterior* Reissner–Nordström–de Sitter space–time region ($r > r_{(0)}$) — one “universe” for $\eta > 0$ and another identical “universe” for $\eta < 0$,

glued together along the single Reissner–Nordström–de Sitter horizon at $r = r_{(0)}$, i.e. $\eta = 0$. Since in the exterior Reissner–Nordström–de Sitter region the coordinates (t, r) exchange their roles, whereupon r becomes timelike, the same is true for the coordinate η in (80), (81), i.e. η is timelike coordinate in both “universes” except at the matching hypersurface $\eta = 0$. The latter is directly seen upon transforming (80) into diagonal form $((v, \eta) \rightarrow (\xi, \eta)$ with $d\xi = dv - \frac{d\eta}{\tilde{A}_\Lambda(\eta)}$):

$$ds^2 = -\frac{d\eta^2}{\tilde{A}_\Lambda(\eta)} + \tilde{A}_\Lambda(\eta)d\xi^2 + \tilde{r}^2(\eta)(d\theta^2 + \sin^2\theta d\varphi^2) \tag{83}$$

with $\tilde{A}_\Lambda(\eta)$ as in (81) and $\eta \in (-\infty, +\infty)$.

Now, repeating the same steps as in the derivation of the wormhole connecting two nonsingular black holes in the first part of the present section (Eqs. (65)–(73)) we obtain that the metric (80), (81) is a solution of the Einstein equations (76) with the LL-brane stress–energy tensor *included*. Here again $T_{\mu\nu}^{(\text{brane})}$ has the same form as in (68), (69), whereas the matching of the coefficients in front of the delta-functions $\delta(\eta)$ on both sides of (76) yield (cf. Eqs. (72) above):

$$\frac{\Lambda}{3}r_{(0)} + \frac{Q^2}{r_{(0)}^3} - \frac{m}{r_{(0)}^2} = 4\pi T\sqrt{b_0}, \quad r_{(0)} = \frac{\sqrt{b_0}}{2\pi T}, \tag{84}$$

where $r_{(0)} \equiv r_{(0)}(m, Q^2, \Lambda)$ is the single Reissner–Nordström–de Sitter horizon radius (79). Equations (84) determine m and Q^2 as functions of the bare cosmological constant Λ , and of T (the LL-brane dynamical tension) and the free parameter b_0 of the LL-brane dynamics (cf. Appendix).

The physical meaning of the space–time manifold with the metric (80), (81) or (83) can be more clearly seen upon making the following change of the time coordinate $\eta \rightarrow \bar{\eta}$ with

$$d\bar{\eta} = \frac{d\eta}{\sqrt{\tilde{A}_\Lambda(\eta)}}, \quad \bar{\eta} \simeq \frac{1}{2\sqrt{A_0}} \text{sign}(\eta)\sqrt{|\eta|} \quad \text{for small } \eta, \tag{85}$$

where $A_0 \equiv -\partial_r A_\Lambda|_{r=r_{(0)}} > 0$, such that (83) becomes

$$ds^2 = -d\bar{\eta}^2 + \hat{A}_\Lambda(\bar{\eta})d\xi^2 + \hat{r}^2(\bar{\eta})(d\theta^2 + \sin^2\theta d\varphi^2), \tag{86}$$

$$\hat{A}_\Lambda(\bar{\eta}) = \tilde{A}_\Lambda(\eta(\bar{\eta})) = -A_\Lambda(r_{(0)} + |\eta(\bar{\eta})|), \quad \hat{r}(\bar{\eta}) = r_{(0)} + |\eta(\bar{\eta})|, \tag{87}$$

(here A_Λ and $r_{(0)}$ are the same as in (78), (79)). The coordinate singularity of the metric (83) at $\eta = 0$ or of the metric (86) at $\bar{\eta} = 0$ is purely artificial one as it is absent in the Eddington–Finkelstein form (80).

The new timelike coordinate $\bar{\eta}$ varies again from $-\infty$ to $+\infty$. The metric (86) is a special case of the anisotropic Kantowski–Sachs cosmology,⁵⁷ where the three spatial dimensions are having a dynamical evolution. In the present case the two spherical angular dimensions are having the same dynamical evolution, which produces a bounce from a nonzero value, while the additional spatial dimension ξ exhibits a different bounce behavior, namely bouncing from zero size although this

is not an intrinsic space–time singularity as explained above. A distinguishing feature of the present solution is that the universe evolving from negative time ($\bar{\eta} < 0$) undergoes contraction until $\bar{\eta} = 0$, but then proceeds for $\bar{\eta} > 0$ with an expansion instead of contraction.

8. Discussion and Conclusions

In the present paper we have explicitly constructed a nonsingular black hole as a self-consistent solution of the equations of motion derived from a well-defined Lagrangian action principle for a bulk Einstein–Maxwell–Kalb–Ramond system coupled to a codimension-one charged *lightlike* p -brane with dynamical (variable) tension (LL-brane). We stress on the fact that the latter is described by a manifestly reparametrization-invariant worldvolume action of either Polyakov-type or Nambu–Goto-type which is significantly different from the ordinary Nambu–Goto p -brane action. The crucial point in our construction is that LL-brane turns out to be the proper gravitational and charge source in the Einstein–Maxwell–Kalb–Ramond equations of motion needed to generate a self-consistent solution describing *nonsingular* black hole. The latter consists of de Sitter interior region and exterior Reissner–Nordström region glued together along their common horizon, which is the inner horizon from the Reissner–Nordström side. The matching horizon is automatically occupied by the LL-brane as a result of its worldvolume Lagrangian dynamics, which also generates the cosmological constant in the interior region and uniquely determines the mass and charge parameters of the exterior region. In particular, Eq. (61) tells us that the size, mass and charge of the nonsingular black hole might be very small for large β , i.e. provided the LL-brane is strongly coupled to the bulk Kalb–Ramond gauge field.

Further, we have constructed a self-consistent *wormhole* solution of Einstein–Maxwell system coupled to electrically neutral LL-brane, which describes two regular black hole space–time regions matched along the “throat” which is their common horizon. The two black hole “universes” are identical copies of the exterior Reissner–Nordström region above the inner Reissner–Nordström horizon — the “throat,” which is now occupied by the LL-brane. The corresponding mass and charge parameters of the black hole “universes” are explicitly determined by the dynamical LL-brane tension. This also provides an explicit example of Misner–Wheeler “charge without charge” phenomenon. Provided a sufficiently large bare cosmological constant is added, we have shown that the above wormhole solution connecting two nonsingular black holes can be transformed into a cosmological bouncing solution of Kantowski–Sachs type.

Mechanisms for singularity avoidance in regular black hole solutions have been thoroughly analyzed in Refs. 53 and 54. Similarly to the previously derived nonsingular black holes, the existence and consistency of the presently proposed regular black hole solution (Sec. 5) can be attributed to the topology change in the structure of the corresponding spacelike slices.

At this point the issue of stability remains an open question. Let us note that the latter is greatly affected by the nature of the matching shell dynamics, see e.g. Ref. 68. It was shown there that if the *timelike* shell separating a Schwarzschild region from a de Sitter region obeys a “domain wall” equation of state, then the equation of motion of the wall corresponds to the equation of motion of a particle in a potential with only one stationary point which is a maximum. This does not tell us a lot about the LL-brane case since the latter are not ordinary domain walls. In fact, when additional matter is added on the wall, the wall potential may acquire a minimum.⁷³ In addition to this, in case when the surface tension becomes dynamically zero at certain shell radius, the potential of the shell displays an attractive behavior towards that same radius.⁷⁴ Another argument in favor of the stability of the solution in the case of Reissner–Nordström exterior region and equality of the radial pressures on both sides of the shell (i.e. zero surface tension) is the following fact. If the radius of the shell is lowered, the de Sitter pressure becomes bigger than the Reissner–Nordström pressure. The opposite takes place when the shell radius is increased. All this indicates a “mechanical” argument for stability. There is, however, a “kinematical” argument against stability since a perturbation that makes a shell radius bigger than the de Sitter horizon will force the shell to expand to infinity leading to the creation of a child universe.⁶⁸

It is not clear how the above arguments can be applied to the case of LL-branes since the LL-brane worldvolume dynamics forces the latter to locate itself automatically on certain horizon of the embedding space–time metrics.

The regular black hole solution via LL-brane constructed above may still be interesting even if it turns out to be unstable, since the instability (if it exists) appears to be towards formation of a child universe. Furthermore, unstable solutions may play an important role in quantum effects like sphaleron solutions⁷⁵ in Glashow–Weinberg–Salam model. All these issues should be a subject of further studies.

Finally, let us mention that quantum effects may have significant impact on the fate of black hole singularities, namely, removing them altogether (Refs. 16–19, 76, 77). It would be very interesting to study the quantization of the above Einstein–Maxwell–Kalb–Ramond system interacting with charged LL-brane where black hole singularities have been removed already at classical level by the presence of the LL-brane alone.

Appendix A. Nambu–Goto-Type Worldvolume Formulation of Lightlike Branes

Lets us consider the *dual* Nambu–Goto-type worldvolume action of the LL-brane:

$$S_{\text{NG}} = - \int d^{p+1} \sigma T \sqrt{\left| \det \left\| g_{ab} - \epsilon \frac{1}{T^2} \partial_a u \partial_b u \right\| \right|}, \quad \epsilon = \pm 1, \quad (\text{A.1})$$

where g_{ab} indicates the induced metric on the worldvolume (7). The choice of the sign in (A.1) does not have physical effect because of the nondynamical nature of the u -field.

The corresponding equations of motion w.r.t. X^μ , u and T read

$$\partial_a \left(T \sqrt{|\tilde{g}|} \tilde{g}^{ab} \partial_b X^\mu \right) + T \sqrt{|\tilde{g}|} \tilde{g}^{ab} \partial_a X^\lambda \partial_b X^\nu \Gamma_{\lambda\nu}^\mu = 0, \tag{A.2}$$

$$\partial_a \left(\frac{1}{T} \sqrt{|\tilde{g}|} \tilde{g}^{ab} \partial_b u \right) = 0, \tag{A.3}$$

$$T^2 + \epsilon \tilde{g}^{ab} \partial_a u \partial_b u = 0, \tag{A.4}$$

where we have introduced the convenient notations

$$\tilde{g}_{ab} \equiv g_{ab} - \epsilon \frac{1}{T^2} \partial_a u \partial_b u, \quad \tilde{g} \equiv \det \|\tilde{g}_{ab}\|, \tag{A.5}$$

and \tilde{g}^{ab} is the inverse matrix w.r.t. \tilde{g}_{ab} .

From the definition (A.5) and Eq. (A.4) one easily finds that the induced metric on the worldvolume is singular on-shell (cf. Eq. (23) above):

$$g_{ab} (\tilde{g}^{bc} \partial_c u) = 0 \tag{A.6}$$

exhibiting the lightlike nature of the p -brane described by (A.1).

Similarly to the treatment of the LL-brane dynamics within the Polyakov-type formulation (Sec. 3 above) we can choose the following gauge-fixing of worldvolume reparametrization invariance (cf. Eqs. (27)):

$$\tilde{g}_{0i} = 0 \quad (i = 1, \dots, p), \quad \tilde{g}_{00} = -\epsilon b_0, \quad b_0 = \text{const} > 0. \tag{A.7}$$

Also, we will use a natural ansatz

$$u = u(\tau), \quad \text{i.e.} \quad \partial_i u = 0, \tag{A.8}$$

which means that we choose the lightlike direction in Eq. (A.6) to coincide with the brane proper-time direction on the worldvolume (this is analogous to the ansatz (28) within the Polyakov-type formulation). With (A.7), (A.8) Eq. (A.3) implies (cf. Eq. (29) above):

$$\partial_0 g^{(p)} = 0 \quad \text{where} \quad g^{(p)} \equiv \det \|g_{ij}\|, \tag{A.9}$$

with g_{ij} being the spacelike part of the induced metric (7).

Taking into account (A.7)–(A.9), the equations of motion (A.2)–(A.4) (or, equivalently, (A.6)) reduce to (cf. Eqs. (30)–(34)):

$$g_{00} \equiv \dot{X}^\mu G_{\mu\nu} \dot{X}^\nu = 0, \quad g_{0i} \equiv \dot{X}^\mu G_{\mu\nu} \partial_i X^\nu = 0, \quad \partial_i T = 0, \tag{A.10}$$

$$\begin{aligned} & -\partial_0 (T \partial_0 X^\mu) - \frac{T \epsilon b_0}{\sqrt{g^{(p)}}} \partial_i \left(\sqrt{g^{(p)}} g^{ij} \partial_j X^\mu \right) \\ & + T (\partial_0 X^\nu \partial_0 X^\lambda - \epsilon b_0 g^{kl} \partial_k X^\nu \partial_l X^\lambda) \Gamma_{\nu\lambda}^\mu = 0. \end{aligned} \tag{A.11}$$

The LL-brane stress–energy tensor derived from the Nambu–Goto-type action (A.1) reads (cf. (18)):

$$T_{(\text{brane})}^{\mu\nu} = - \int d^{p+1}\sigma \frac{\delta^{(D)}(x - X(\sigma))}{\sqrt{-G}} T \sqrt{|\tilde{g}|} \tilde{g}^{ab} \partial_a X^\mu \partial_b X^\nu, \quad (\text{A.12})$$

where the short-hand notation (A.5) is used. For the embedding (36) and taking into account (A.7) we obtain

$$T_{(\text{brane})}^{\mu\nu} = S^{\mu\nu} \delta(r - r_0), \quad (\text{A.13})$$

$$S^{\mu\nu} \equiv \frac{T}{\epsilon \sqrt{b_0}} [\partial_\tau X^\mu \partial_\tau X^\nu - \epsilon b_0 G^{ij} \partial_i X^\mu \partial_j X^\nu]_{v=\tau, r=r_0, \theta^i=\sigma^i},$$

where r_0 is a horizon of the bulk space–time metric. The LL-brane stress–energy tensor (A.13) is identical to the one obtained within the Polyakov-type formulation (Eq. (46) above) with the identifications

$$\chi = T b_0^{\frac{p-1}{2}} \epsilon^{\frac{p-2}{2}}, \quad 2a_0 = \epsilon b_0, \quad (\text{A.14})$$

where χ , a_0 and b_0 are the same as in (10), (31) and (A.7), respectively.

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